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**Weak convergence of Markovian random evolution
in a multidimensional space.**

Abstract. We study Markovian symmetry and non-symmetry random evolutions in \mathbf{R}^n . Weak convergence of Markovian symmetry random evolution to Wiener process and of Markovian non-symmetry random evolution to a diffusion process with drift is proved using problems of singular perturbation for the generators of evolutions. Relative compactness in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$ of the families of Markovian random evolutions is also shown.

Keywords. Markovian random evolution, symmetry, weak convergence, singular perturbation problem, relative compactness.

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1 Introduction

Markov symmetry random evolutions (MSRE) in spaces of different dimensions were studied in the works of M.Kac [8], M.Pinsky [16], E.Orzinger (e.g., [14, 15]), A.F.Turbin and A.D.Kolesnik (e.g., [9, 10]) (see also [18] for other references). Symmetry in this sense should be regarded as uniform stationary distribution of switching at a symmetrical structure in \mathbf{R}^n , for instance at $n + 1$ -hedron [18], or at a unit sphere [12].

Weak convergence of distributions of MSRE is also studied in some of these works, namely convergence in \mathbf{R}^2 and \mathbf{R}^3 was proved by A.D.Kolesnik in [11, 12].

We should note that the problem of weak convergence of random walks (partially, similar to MSRE) was studied by many authors. Among the most interesting works we may point [3, 4, 6, 7]. Large bibliography as for this branch could be found in [3]. The methods, proposed in these works let us solve a wide range of problems connected with convergence of random walks, but do not let to obtain limit process to be averaged by the stationary measure of switching process.

Such averaging may be found in the works of V.V.Anisimov and his students (see [1] and references therein), but here the averaging by the stationary measure is one of the conditions, proposed for the pre-limit process in the corresponding theorem.

In this work MSRE in \mathbf{R}^n is studied using the methods, proposed in [13]. We find a solution of singular perturbation problem for the generator of the evolution and thus the averaging by a stationary measure of switching process is obtained as a corollary of this solution. At the second stage we prove relative compactness of the family of MSRE. This method let us show weak convergence of the process of MSRE to the Wiener process in \mathbf{R}^n .

The difference in the methods may be easily seen by the analysis of the papers [2] and [19], where similar problems are studied.

In the sections 4 and 5 we use the method proposed to prove weak convergence of *Markov non-symmetry random evolution* (MNRE) in \mathbf{R}^n . The distinction of this model is that the limit process is a diffusion process with deterministic drift.

2 Description of MSRE

We study a particle in the space \mathbf{R}^n , that starts at $t = 0$ from the point $x = (x_i, i = \overline{1, n})$. Possible directions of motion are given by the vectors

$$s(\theta) = (\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots, \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}, \\ \sin \theta_1 \dots \sin \theta_{n-2} \sin \theta_{n-1}), \theta_{n-1} \in [0, 2\pi), \theta_i \in [0, \pi), i = \overline{1, n-2}.$$

These vectors have initial point in the center of the unit n -dimensional sphere S_n and the terminal point at its surface. Challenge of every next direction is random and its time is distributed by Poisson. Thus, the switching process is Poisson one with intensity $\lambda = \varepsilon^{-2}$. The velocity of particle's motion is fixed and equals $v = c\varepsilon^{-1}$, where ε - is a small parameter, $\varepsilon \rightarrow 0$ ($\varepsilon > 0$).

Let's define a set $\Theta = \{\theta : s(\theta) \in S_n\}$. The switching Poisson process is $\theta_t^\varepsilon \in \Theta$.

Definition 1: *Markov symmetry random evolution* (MSRE) is the process $\xi_t^\varepsilon \in \mathbf{R}^n$, that is given by:

$$\xi_t^\varepsilon := x + v \int_0^t s(\theta_\tau^\varepsilon) d\tau.$$

Easy to see that when $\varepsilon \rightarrow 0$ ($\varepsilon > 0$) the velocity of the particle and intensity of switching decrease. Our aim is to prove weak convergence of MSRE to the Wiener process when $\varepsilon \rightarrow 0$. The main method is solution of singular perturbation problem for the generator of MSRE. Let's describe this generator.

Two-component Markov process $(\xi_t^\varepsilon, \theta_t^\varepsilon)$ at the test-functions $\varphi(x_1, \dots, x_n; \theta) \in C_0^\infty(\mathbf{R}^n \times \Theta)$ may be described by a generator (see, e.g. [16])

$$L^\varepsilon \varphi(x_1, \dots, x_n; \theta) := \lambda Q\varphi(\cdot; \theta) + vS(\theta)\varphi(x_1, \dots, x_n; \cdot) = \\ \varepsilon^{-2}Q\varphi(\cdot; \theta) + \varepsilon^{-1}cS(\theta)\varphi(x_1, \dots, x_n; \cdot), \quad (1)$$

where

$$S(\theta)\varphi(x_1, \dots, x_n; \cdot) := -(s(\theta), \nabla)\varphi(x_1, \dots, x_n; \cdot),$$

here $\nabla\varphi = (\partial\varphi/\partial x_i, 1 \leq i \leq n)$,

$$Q\varphi(\cdot; \theta) := (\Pi - I)\varphi(\cdot; \theta) := \frac{1}{N} \int_{S_n} \varphi(\cdot; \theta) \mu(d\theta) - \varphi(\cdot; \theta),$$

here $N = (2\pi)^{n/2} \frac{1}{2 \cdot 4 \cdot \dots \cdot (n-2)}$ for even n , and $N = (2\pi)^{(n-1)/2} \frac{2}{3 \cdot 5 \cdot \dots \cdot (n-2)}$ for odd n ; $\mu(d\theta)$ - is the element of volume of the sphere S_n , that is equal to

$$\mu(d\theta) := \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} d\theta_1 \dots d\theta_{n-1}.$$

Using well-known formula

$$\int_0^\pi \sin^{2m} \theta d\theta = 2 \cdot \frac{1 \cdot 2 \cdot \dots \cdot (2m-1)}{2 \cdot 4 \cdot \dots \cdot 2m} \cdot \frac{\pi}{2}, \int_0^\pi \sin^{2m+1} \theta d\theta = 2 \cdot \frac{2 \cdot 4 \cdot \dots \cdot 2m}{1 \cdot 3 \cdot \dots \cdot (2m+1)}$$

we may see

$$\int_{S_n} \mu(d\theta) = N.$$

Operator $\Pi\varphi(\cdot; \theta) := \frac{1}{N} \int_{S_n} \varphi(\cdot; \theta) \mu(d\theta)$ is the projector at the null-space of reducibly-invertible operator Q , because by definition it transfers functions to constants, but constants to itself.

For Π we have:

$$Q\Pi = \Pi Q = 0.$$

Potential operator R_0 may be defined by:

$$R_0 := \Pi - I.$$

This operator has the property:

$$R_0 Q = Q R_0 = I - \Pi,$$

thus it is inverse for Q in the range of Q , but for the function ϕ from null-space of Q we have

$$R_0 \phi = 0.$$

Solution of singular perturbation problem in the series scheme with the small series parameter $\varepsilon \rightarrow 0$ ($\varepsilon > 0$) (see [13]) for reducibly-invertible operator Q and perturbed operator Q_1 consists in the following.

We should find a vector $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ and a vector ψ , that satisfy asymptotic representation

$$[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1]\varphi^\varepsilon = \psi + \varepsilon\theta^\varepsilon$$

with the vector θ^ε , that is uniformly bounded by the norm and such that

$$\|\theta^\varepsilon\| \leq C, \varepsilon \rightarrow 0.$$

The left part of the equation may be rewritten

$$[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = \varepsilon^{-2}Q\varphi + \varepsilon^{-1}[Q\varphi_1 + Q_1\varphi] + [Q\varphi_2 + Q_1\varphi_1] + \varepsilon Q_1\varphi_2.$$

And as soon as it is equal to the right side, we obtain:

$$\begin{cases} Q\varphi = 0, \\ Q\varphi_1 + Q_1\varphi = 0, \\ Q\varphi_2 + Q_1\varphi_1 = \psi, \\ Q_1\varphi_2 = \theta^\varepsilon. \end{cases}$$

From the last equation we may see that the function φ_2 should be smooth enough to provide boundness of $Q_1\varphi_2$. Moreover, from the first equation we see that the function φ may be any function from the null-space of Q and does not depend on the variable that corresponds to the switching process.

An important condition of solvability of this problem is balance condition

$$\Pi Q_1 = 0.$$

This condition means that the function $Q_1\varphi$ belongs to the range of Q , thus we may solve the second equation using the potential operator, that is inverse to Q at its range

$$\varphi_1 = -R_0 Q_1 \varphi. \tag{2}$$

Thus the main problem is to solve the equation

$$Q\varphi_2 = \psi - Q_1\varphi_1 = \psi + Q_1 R_0 Q_1 \varphi.$$

The solvability condition for Q has the view:

$$\Pi Q \Pi \varphi_2 = 0 = \Pi \psi + \Pi Q_1 R_0 Q_1 \Pi \varphi,$$

and we finally obtain

$$\Pi\psi = -\Pi Q_1 R_0 Q_1 \Pi\varphi. \quad (3)$$

For the function φ_2 we obviously have:

$$\varphi_2 = R_0[-\Pi Q_1 R_0 Q_1 \Pi + Q_1 R_0 Q_1]\varphi. \quad (4)$$

Equations (2)-(4) give the solution of singular perturbation problem.

In case of MSRE balance condition has the following view

$$\Pi S(\theta)\mathbf{1}(x) = 0, \quad (5)$$

where $\mathbf{1}(x) = (x_1, \dots, x_n)$. Really, every term under the sign of integral contains either $\int_0^\pi \sin^n \theta \cos \theta d\theta = 0$ or $\int_0^{2\pi} \sin \theta d\theta = 0$.

3 Main result for MSRE

THEOREM 1. MSRE ξ_t^ε , converges weakly to the Wiener process $w(t) := \xi_t^0$ when $\varepsilon \rightarrow 0$:

$$\xi_t^\varepsilon \Rightarrow \xi_t^0,$$

where $\xi_t^0 \in \mathbf{R}^n$ is defined by a generator

$$L^0\varphi(x_1, \dots, x_n) = \frac{c^2}{n}\Delta\varphi(x_1, \dots, x_n), \quad (6)$$

where $\Delta\varphi := \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)\varphi$.

Remark 1: The generator (6) corresponds to the results, obtained in [12] for the spaces \mathbf{R}^2 and \mathbf{R}^3 .

To prove the Theorem we need the following Lemma.

Lemma 1. At the perturbed test-functions

$$\varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varphi(x_1, \dots, x_n) + \varepsilon\varphi_1(x_1, \dots, x_n; \theta) + \varepsilon^2\varphi_2(x_1, \dots, x_n; \theta), \quad (7)$$

that have bounded derivatives of any degree and compact support, the operator L^ε has asymptotic representation

$$L^\varepsilon\varphi^\varepsilon(x_1, \dots, x_n; \theta) = L^0\varphi(x_1, \dots, x_n) + R^\varepsilon(\theta)\varphi(x),$$

$$|R^\varepsilon(\theta)\varphi(x)| \rightarrow 0, \varepsilon \rightarrow 0, \varphi(x) \in C_0^\infty(\mathbf{R}^d),$$

where L^0 is defined in (6), $\varphi_1(x_1, \dots, x_n; \theta)$, $\varphi_2(x_1, \dots, x_n; \theta)$ and $R^\varepsilon(\theta)\varphi(x)$ are the following:

$$L^0\Pi = -c^2\Pi S(\theta)R_0S(\theta)\Pi, \quad (8)$$

$$\varphi_1 = -cR_0S(\theta)\varphi,$$

$$\varphi_2 = c^2R_0S(\theta)R_0S(\theta)\varphi,$$

$$R^\varepsilon(\theta)\varphi = \varepsilon c^3S(\theta)R_0S(\theta)R_0S(\theta)\varphi.$$

Proof. Let's solve singular perturbation problem for the operator (2). To do this, we study this operator at the test-function (7). We have:

$$L^\varepsilon\varphi^\varepsilon(x_1, \dots, x_n; \theta) = [\varepsilon^{-2}Q + \varepsilon^{-1}cS(\theta)][\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2] = \varepsilon^{-2}Q\varphi +$$

$$\varepsilon^{-1}[Q\varphi_1 + cS(\theta)\varphi] + [Q\varphi_2 + cS(\theta)\varphi_1] + \varepsilon cS(\theta)\varphi_2.$$

Thus, we obtain the following equations

$$\begin{cases} Q\varphi = 0, \\ Q\varphi_1 + cS(\theta)\varphi = 0, \\ L^0\varphi = Q\varphi_2 + cS(\theta)\varphi_1, \\ R^\varepsilon\varphi(\theta) = \varepsilon cS(\theta)\varphi_2. \end{cases} \quad (9)$$

From the first equation we see that $\varphi(x_1, \dots, x_n)$ belongs to the null-space of Q . From the balance condition (5) easy to see that $S(\theta)\varphi$ belongs to the range of Q , thus from the second equation of the system (9) we obtain

$$\varphi_1 = -cR_0S(\theta)\varphi.$$

By substitution into the third equation and using the solvability condition we have:

$$L^0\Pi\varphi + c^2\Pi S(\theta)R_0S(\theta)\Pi\varphi = 0.$$

From the last equation of (5):

$$R^\varepsilon(\theta)\varphi(x) = \varepsilon c^3S(\theta)R_0S(\theta)R_0S(\theta)\varphi(x) \rightarrow 0 \text{ when } \varepsilon \rightarrow 0, \varphi(x) \in C_0^\infty(\mathbf{R}^n).$$

Let's find the generator of the limit process L^0 by the formula (8):

$$L^0\varphi = c^2\Pi S(\theta)(I - \Pi)S(\theta)\Pi\varphi = c^2\Pi S^2(\theta)\Pi\varphi - c^2\Pi S(\theta)\Pi S(\theta)\Pi\varphi.$$

The last term equals to 0 by the balance condition (5). Thus, finally:

$$L^0 = c^2 \Pi S^2(\theta), \quad (10)$$

or

$$L^0 = \frac{c^2}{N} \int_{S_n} S^2(\theta) \mu(d\theta).$$

Let's calculate the integral:

$$\begin{aligned} & \frac{c^2}{N} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \left[\cos^2 \theta_1 \frac{\partial^2}{\partial x_1^2} + \sin^2 \theta_1 \cos^2 \theta_2 \frac{\partial^2}{\partial x_2^2} + \dots + \right. \\ & \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \cos^2 \theta_{n-1} \frac{\partial^2}{\partial x_{n-1}^2} + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin^2 \theta_{n-1} \frac{\partial^2}{\partial x_n^2} + \\ & \left\{ \sin \theta_1 \cos \theta_1 \cos \theta_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \dots + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin \theta_{n-1} \times \right. \\ & \left. \left. \cos \theta_{n-1} \frac{\partial^2}{\partial x_{n-1} \partial x_n} \right\} \right] \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} d\theta_1 \dots d\theta_{n-1} = \\ & \frac{c^2}{N} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \left[\cos^2 \theta_1 \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} \frac{\partial^2}{\partial x_1^2} + \right. \\ & \sin^n \theta_1 \cos^2 \theta_2 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} \frac{\partial^2}{\partial x_2^2} + \dots + \\ & \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} \cos^2 \theta_{n-1} \frac{\partial^2}{\partial x_{n-1}^2} + \\ & \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^2 \theta_{n-2} \sin^2 \theta_{n-1} \frac{\partial^2}{\partial x_n^2} + \\ & \left\{ \sin^{n-1} \theta_1 \cos \theta_1 \cos \theta_2 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} \frac{\partial^2}{\partial x_1 \partial x_2} + \dots + \right. \\ & \left. \left. \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} \sin \theta_{n-1} \cos \theta_{n-1} \frac{\partial^2}{\partial x_{n-1} \partial x_n} \right\} \right] d\theta_1 \dots d\theta_{n-1} = \end{aligned}$$

$$\begin{aligned}
& \frac{c^2}{N} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \left[(\sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} - \sin^n \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2}) \frac{\partial^2}{\partial x_1^2} + \right. \\
& (\sin^n \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} - \sin^n \theta_1 \sin^{n-1} \theta_2 \sin^{n-4} \theta_3 \dots \sin \theta_{n-2}) \frac{\partial^2}{\partial x_2^2} + \dots + \\
& (\sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} - \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} \sin^2 \theta_{n-1}) \frac{\partial^2}{\partial x_{n-1}^2} + \\
& \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} \sin^2 \theta_{n-1} \frac{\partial^2}{\partial x_n^2} + \\
& \left. \left\{ \sin^{n-1} \theta_1 \cos \theta_1 \cos \theta_2 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} \frac{\partial^2}{\partial x_1 \partial x_2} + \dots + \right. \right. \\
& \left. \left. \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^3 \theta_{n-2} \sin \theta_{n-1} \cos \theta_{n-1} \frac{\partial^2}{\partial x_{n-1} \partial x_n} \right\} \right] d\theta_1 \dots d\theta_{n-1}.
\end{aligned}$$

Every term in braces has a multiplier of the type $\int_0^\pi \sin^n \theta \cos \theta d\theta = 0$ or $\int_0^{2\pi} \sin \theta \cos \theta d\theta = 0$, thus the corresponding integral equals to 0.

Integration of every correlation in parentheses gives

$$\begin{aligned}
& \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} (\sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^{n-k+2} \theta_{k-1} \sin^{n-k-1} \theta_k \sin^{n-k-2} \theta_{k+1} \dots \sin \theta_{n-2} - \\
& \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^{n-k+2} \theta_{k-1} \sin^{n-k+1} \theta_k \sin^{n-k-2} \theta_{k+1} \dots \sin \theta_{n-2}) d\theta_1 \dots d\theta_{n-1} = \\
& N \left(\frac{(2m-1)(2m-2)(2m-3) \dots (2m-k+1)}{2m(2m-1)(2m-2) \dots (2m-k+2)} - \right. \\
& \left. \frac{(2m-1)(2m-2)(2m-3) \dots (2m-k)}{2m(2m-1)(2m-2) \dots (2m-k+1)} \right) = \\
& N \left(\frac{2m-k+1}{2m} - \frac{2m-k}{2m} \right), \text{ if } n = 2m \\
& \text{or } N \left(\frac{(2m)(2m-1)(2m-2) \dots (2m-k+2)}{(2m+1)2m(2m-1) \dots (2m-k+3)} - \right. \\
& \left. \frac{2m(2m-1)(2m-2) \dots (2m-k+1)}{(2m+1)2m(2m-1) \dots (2m-k+2)} \right) =
\end{aligned}$$

$$N \left(\frac{2m - k + 2}{2m + 1} - \frac{2m - k + 1}{2m + 1} \right), \text{ if } n = 2m + 1$$

$$= \frac{N}{n}.$$

Finally, we have:

$$L^0 \varphi(x_1, \dots, x_n) = \frac{c^2}{n} \Delta \varphi(x_1, \dots, x_n).$$

Lemma is proved.

Proof of Theorem 1. In 1 we proved that $L^\varepsilon \varphi^\varepsilon \Rightarrow L^0 \varphi$ at the class of functions $C_0^\infty(\mathbf{R}^n \times \Theta)$ when $\varepsilon \rightarrow 0$. To prove the weak convergence we need to show the relative compactness of the family $(\xi_t^\varepsilon, \theta_t^\varepsilon)$ in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$. To do this we use the methods proposed in [5, 13, 20]. Let's formulate Corollary 6.1 from [13] (see also Theorem 6.4 in [13]) as a Lemma.

Lemma 2. Let the generators $L^\varepsilon, \varepsilon > 0$ satisfy the inequalities

$$|L^\varepsilon \varphi(u)| < C_\varphi$$

for any real-valued non-negative function $\varphi \in C_0^\infty(\mathbf{R}^n \times \Theta)$, where the constant C_φ depends on the norm of φ , and for $\varphi_0(u) = \sqrt{1 + u^2}$,

$$L^\varepsilon \varphi_0(u) \leq C_l \varphi_0(u), |u| \leq l,$$

where the constant C_l depends on the function φ_0 , but do not depend on $\varepsilon > 0$.

Then the family $(\xi_t^\varepsilon, \theta_t^\varepsilon), t \geq 0, \varepsilon > 0$ is relatively compact in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$.

Let's study (2) at the test-function $\varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varphi(x_1, \dots, x_n) + \varepsilon \varphi_1(x_1, \dots, x_n; \theta)$, where $\varphi_1(x_1, \dots, x_n; \theta) = -c R_0 S(\theta) \varphi(x_1, \dots, x_n)$.

We have:

$$L^\varepsilon \varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varepsilon^{-2} Q \varphi(x_1, \dots, x_n) + \varepsilon^{-1} [Q \varphi_1 + c S(\theta) \varphi(x_1, \dots, x_n)] +$$

$$c S(\theta) \varphi_1(x_1, \dots, x_n; \theta).$$

It follows from (9) that two first terms equal to 0. Let's estimate the last term:

$$c S(\theta) \varphi_1(x_1, \dots, x_n; \theta) = c^2 S(\theta) R_0 S(\theta) \varphi(x_1, \dots, x_n) = c^2 S^2(\theta) \varphi(x_1, \dots, x_n) =$$

$$\begin{aligned}
& c^2 \left[\cos^2 \theta_1 \frac{\partial^2}{\partial x_1^2} + \sin^2 \theta_1 \cos^2 \theta_2 \frac{\partial^2}{\partial x_2^2} + \dots + \right. \\
& \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \cos^2 \theta_{n-1} \frac{\partial^2}{\partial x_{n-1}^2} + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin^2 \theta_{n-1} \frac{\partial^2}{\partial x_n^2} + \\
& \left. \left\{ \sin \theta_1 \cos \theta_1 \cos \theta_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \dots + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin \theta_{n-1} \times \right. \right. \\
& \left. \left. \cos \theta_{n-1} \frac{\partial^2}{\partial x_{n-1} \partial x_n} \right\} \right] \varphi(x_1, \dots, x_n) \leq C_{1,\varphi},
\end{aligned}$$

as soon as all the constants, functions and their derivatives are bounded.

We also have from (9):

$$L^\varepsilon \varphi^\varepsilon = L^\varepsilon \varphi + \varepsilon L^\varepsilon \varphi_1 = L^\varepsilon \varphi + \varepsilon c L^\varepsilon R_0 S(\theta) \varphi.$$

Thus,

$$L^\varepsilon \varphi = L^\varepsilon \varphi^\varepsilon - \varepsilon c L^\varepsilon R_0 S(\theta) \varphi \leq C_{1,\varphi} - \varepsilon C_{2,\varphi} < C_\varphi$$

for small ε .

To prove the second condition, it's enough to use the properties of the function $\varphi_0(u) = \sqrt{1+u^2}$, namely:

$$|\varphi'_0(u)| \leq 1 \leq \varphi_0(u), |\varphi''_0(u)| \leq \varphi_0(u).$$

So, the proof of the second condition is similar to the previous reasoning.

Thus, the family $(\xi_t^\varepsilon, \theta_t^\varepsilon)$ is relatively compact in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$.

Now we may use the following theorem (Theorem 6.6 from [13]).

THEOREM 2. Let random evolution with Markov switching $(\xi^\varepsilon(t), x^\varepsilon(t)) \in \mathbf{D}_{\mathbf{R}^n \times E}[0, \infty)$ satisfies the conditions:

C1: The family of processes $(\xi^\varepsilon(t), t \geq 0, \varepsilon > 0)$ is relatively compact.

C2: There exists a family of test-functions $\varphi^\varepsilon(u, x) \in C_0^3(\mathbf{R}^n \times E)$ such that

$$\lim_{\varepsilon \rightarrow 0} \varphi^\varepsilon(u, x) = \varphi(u)$$

uniformly by u, x .

C3: The following uniform convergence is true

$$\lim_{\varepsilon \rightarrow 0} L^\varepsilon \varphi^\varepsilon(u, x) = L\varphi(u)$$

uniformly by u, x .

The family $L^\varepsilon \varphi^\varepsilon, \varepsilon > 0$ is uniformly bounded, moreover $L^\varepsilon \varphi^\varepsilon$ and $L\varphi$ belong to $C(\mathbf{R}^n \times E)$.

C4: Convergence by probability of initial values

$$\xi^\varepsilon(0) \rightarrow \widehat{\xi}(0), \varepsilon \rightarrow 0,$$

and

$$\sup_{\varepsilon > 0} \mathbf{E}|\xi^\varepsilon(0)| \leq C < +\infty$$

is true.

Then we have the weak convergence

$$\xi^\varepsilon(t) \Rightarrow \widehat{\xi}(t), \varepsilon \rightarrow 0.$$

According to Theorem 2, we may confirm the weak convergence in $\mathbf{D}_{\mathbf{R}^n}[0, \infty)$

$$\xi_t^\varepsilon \Rightarrow \xi_t^0.$$

Really, all the conditions are satisfied. Namely, the family of processes is relatively compact, the generators at the test-functions from the class $C_0^\infty(\mathbf{R}^n \times \Theta)$ converge, initial conditions for the limit and pre-limit processes are equal.

Theorem is proved.

4 Description of MNRE

In \mathbf{R}^n we study the same particle as in the Section 2, but its velocity is equal to $v(\theta) = c(\theta)\varepsilon^{-1} + c_1(\theta)$, where $\varepsilon \rightarrow 0$ ($\varepsilon > 0$) is the small parameter, functions $c(\theta), c_1(\theta)$ are bounded.

Definition 2: *Markov non-symmetry random evolution* (MNRE) is the process $\widetilde{\xi}_t^\varepsilon \in \mathbf{R}^n$, of the following view:

$$\widetilde{\xi}_t^\varepsilon := x + \int_0^t v(\theta_\tau^\varepsilon) s(\theta_\tau^\varepsilon) d\tau.$$

Our aim is to prove the weak convergence of MNRE to a diffusion process with drift when $\varepsilon \rightarrow 0$.

Two-component Markov process $(\tilde{\xi}_t^\varepsilon, \theta_t^\varepsilon)$ at the test-functions $\varphi(x_1, \dots, x_n; \theta) \in C_0^\infty(\mathbf{R}^n \times \Theta)$ is defined by a generator (see, e.g. [16])

$$\begin{aligned} L^\varepsilon \varphi(x_1, \dots, x_n; \theta) &:= \lambda Q \varphi(\cdot; \theta) + v(\theta) S(\theta) \varphi(x_1, \dots, x_n; \cdot) = \\ &\varepsilon^{-2} Q \varphi(\cdot; \theta) + \varepsilon^{-1} c(\theta) S(\theta) \varphi(x_1, \dots, x_n; \cdot) + c_1(\theta) S(\theta) \varphi(x_1, \dots, x_n; \cdot), \end{aligned} \quad (11)$$

where

$$S(\theta) \varphi(x_1, \dots, x_n; \cdot) := -(s(\theta), \nabla) \varphi(x_1, \dots, x_n; \cdot).$$

An important condition that allows to confirm weak convergence is balance condition

$$\Pi c(\theta) S(\theta) = 0. \quad (12)$$

Remark 2: This is the last condition that defines symmetry or non-symmetry of the process. In case of MSRE the balance condition (12) is true, and $c_1(\theta) \equiv 0$. In case of MNRE non-symmetry of the process is caused by the condition:

$$\Pi c_1(\theta) S(\theta) = (d, \nabla) \neq 0. \quad (13)$$

Example 1: Condition (12) may be satisfied for different functions $c(\theta)$. Namely, in case of MSRE $c(\theta) = c = \text{const}$. Then every term under the integral contains either $\int_0^\pi \sin^n \theta \cos \theta d\theta = 0$ or $\int_0^{2\pi} \sin \theta d\theta = 0$.

Another variant of the function $c(\theta)$, is $c(\theta) = \sin \theta_1$. Really, under the integral we obtain terms, analogical to the previous ones. We note that the dimension of the space in this case should be more than 2, because in \mathbf{R}^2 this function does not preserve the symmetry.

Example 2: Non-symmetry condition (13) is also satisfied for different functions $c_1(\theta)$. For example, in \mathbf{R}^2 for $c_1(\theta) = \sin \theta$ we obtain:

$$\frac{1}{2\pi} \int_0^{2\pi} \sin \theta \left[\cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \right] d\theta = \frac{1}{2} \frac{\partial}{\partial x_2}.$$

Another example is the function in \mathbf{R}^n :

$$c_1(\theta) = \begin{cases} c_1, \theta_{n-1} \in [\pi, 2\pi), \\ 0, \theta_{n-1} \in [0, \pi). \end{cases}$$

Again, all terms under the integral, except the last one contain $\int_0^\pi \sin^n \theta \cos \theta d\theta = 0$, so only one term is not trivial

$$\frac{1}{N} c_1 \int_0^\pi \dots \int_0^\pi \int_\pi^{2\pi} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \dots \sin^2 \theta_{n-1} \sin \theta_{n-1} d\theta_1 \dots d\theta_{n-1} \frac{\partial}{\partial x_n}.$$

By simple calculations, we obtain:

$$\Pi c_1(\theta) S(\theta) = \begin{cases} -\frac{c_1}{2} \frac{3 \cdot 5 \dots (n-2)}{2 \cdot 4 \dots (n-1)} \frac{\partial}{\partial x_n}, n = 2m + 1, \\ -\frac{c_1}{\pi} [1 \cdot 2 \cdot \dots (n-2)] \frac{\partial}{\partial x_n}, n = 2m. \end{cases}$$

Thus we have a wide range of functions that preserve or, on the contrary, do not preserve symmetry. So, we may define the velocity of random evolution in different ways.

5 Main result for MNRE

THEOREM 3. MNRE $\tilde{\xi}_t^\varepsilon$, converges weakly to the process $\tilde{\xi}_t^0$ when $\varepsilon \rightarrow 0$:

$$\tilde{\xi}_t^\varepsilon \Rightarrow \tilde{\xi}_t^0.$$

The limit process $\tilde{\xi}_t^0 \in \mathbf{R}^n$ is defined by a generator

$$L^0 \varphi(x_1, \dots, x_n) = (d, \nabla) \varphi(x_1, \dots, x_n) + (\sigma^2, \Delta) \varphi(x_1, \dots, x_n), \quad (14)$$

where $\Delta \varphi(x_1, \dots, x_n) := \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right) \varphi(x_1, \dots, x_n)$, $(d, \nabla) := -\frac{1}{N} \times \int_{S_n} c_1(\theta) (s(\theta), \nabla) \mu(d\theta)$, $(\sigma^2, \Delta) := \frac{1}{N} \int_{S_n} c^2(\theta) (s(\theta), \nabla)^2 \mu(d\theta)$.

To prove the Theorem, we need the following Lemma.

Lemma 3. At the perturbed test-functions

$$\varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varphi(x_1, \dots, x_n) + \varepsilon \varphi_1(x_1, \dots, x_n; \theta) + \varepsilon^2 \varphi_2(x_1, \dots, x_n; \theta), \quad (15)$$

that have bounded derivatives of any degree and compact support, the operator L^ε has asymptotic representation

$$L^\varepsilon \varphi^\varepsilon(x_1, \dots, x_n; \theta) = L^0 \varphi(x_1, \dots, x_n) + R^\varepsilon(\theta) \varphi(x),$$

$$|R^\varepsilon(\theta)\varphi(x)| \rightarrow 0, \varepsilon \rightarrow 0, \varphi(x) \in C^\infty(\mathbf{R}^d),$$

where L^0 is defined in (14), $\varphi_1(x_1, \dots, x_n; \theta)$, $\varphi_2(x_1, \dots, x_n; \theta)$ and $R^\varepsilon(\theta)\varphi(x)$ are defined by equalities:

$$L^0\Pi = -\Pi c(\theta)S(\theta)R_0c(\theta)S(\theta)\Pi + \Pi c_1(\theta)S(\theta)\Pi\varphi, \quad (16)$$

$$\varphi_1 = -R_0c(\theta)S(\theta)\varphi,$$

$$\varphi_2 = R_0c(\theta)S(\theta)R_0c(\theta)S(\theta)\varphi,$$

$$R^\varepsilon(\theta)\varphi = \{\varepsilon[c(\theta)S(\theta)R_0c(\theta)S(\theta)R_0c(\theta)S(\theta) + c_1(\theta)S(\theta)R_0c(\theta)S(\theta)] + \varepsilon^2 c_1(\theta)S(\theta)R_0c(\theta)S(\theta)R_0c(\theta)S(\theta)\} \varphi.$$

Proof. We solve singular perturbation problem for the operator (4). To do this, we study this operator at the test-function (15). So, we have:

$$\begin{aligned} L^\varepsilon\varphi^\varepsilon(x_1, \dots, x_n; \theta) &= [\varepsilon^{-2}Q + \varepsilon^{-1}c(\theta)S(\theta) + c_1(\theta)S(\theta)][\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2] = \\ &= \varepsilon^{-2}Q\varphi + \varepsilon^{-1}[Q\varphi_1 + c(\theta)S(\theta)\varphi] + [Q\varphi_2 + c(\theta)S(\theta)\varphi_1 + c_1(\theta)S(\theta)\varphi] + \\ &\quad \varepsilon[c(\theta)S(\theta)\varphi_2 + c_1(\theta)S(\theta)\varphi_1] + \varepsilon^2 c_1(\theta)S(\theta)\varphi_2. \end{aligned}$$

Thus, we obtain the following equations:

$$\begin{cases} Q\varphi = 0, \\ Q\varphi_1 + c(\theta)S(\theta)\varphi = 0, \\ L^0\varphi = Q\varphi_2 + c(\theta)S(\theta)\varphi_1 + c_1(\theta)S(\theta)\varphi, \\ R^\varepsilon\varphi(\theta) = \varepsilon[c(\theta)S(\theta)\varphi_2 + c_1(\theta)S(\theta)\varphi_1] + \varepsilon^2 c_1(\theta)S(\theta)\varphi_2. \end{cases} \quad (17)$$

According to the first one $\varphi(x_1, \dots, x_n)$ belongs to the null-space of Q . From the balance condition (12) we see that $c(\theta)S(\theta)\varphi$ belongs to the range of Q , thus from the second equation of (17) we have

$$\varphi_1 = -R_0c(\theta)S(\theta)\varphi.$$

By substitution into the third equation, and using the solvability condition, we obtain:

$$L^0\Pi\varphi + \Pi c(\theta)S(\theta)R_0c(\theta)S(\theta)\Pi\varphi - \Pi c_1(\theta)S(\theta)\Pi\varphi = 0.$$

From the last equation:

$$R^\varepsilon(\theta)\varphi(x) = \{\varepsilon[c(\theta)S(\theta)R_0c(\theta)S(\theta)R_0c(\theta)S(\theta) + c_1(\theta)S(\theta)R_0c(\theta)S(\theta)] +$$

$\varepsilon^2 c_1(\theta) S(\theta) R_0 c(\theta) S(\theta) R_0 c(\theta) S(\theta) \} \varphi(x) \rightarrow 0$ при $\varepsilon \rightarrow 0, \varphi(x) \in C_0^\infty(\mathbf{R}^n)$.

Let's calculate the operator L^0 by the formula (16):

$$L^0 \varphi = \Pi c(\theta) S(\theta) (I - \Pi) c(\theta) S(\theta) \Pi \varphi + \Pi c_1(\theta) S(\theta) \Pi \varphi = \Pi c^2(\theta) S^2(\theta) \Pi \varphi - \\ \Pi c(\theta) S(\theta) \Pi c(\theta) S(\theta) \Pi \varphi + \Pi c_1(\theta) S(\theta) \Pi \varphi.$$

The second term equals 0 by the balance condition (12), the last one is not equal to 0 by (13). Thus, we finally have:

$$L^0 = \Pi c^2(\theta) S^2(\theta) + \Pi c_1(\theta) S(\theta). \quad (18)$$

Using the view of $S(\theta)$, we may write:

$$\Pi c_1(\theta) S(\theta) = -\frac{1}{N} \int_{S_n} c_1(\theta) (s(\theta), \nabla) \mu(d\theta) =: (d, \nabla),$$

$$\Pi c^2(\theta) S^2(\theta) = \frac{1}{N} \int_{S_n} c^2(\theta) (s(\theta), \nabla)^2 \mu(d\theta) =: (\sigma^2, \Delta).$$

Lemma is proved.

Proof of theorem 3. We proved in Lemma 3 that $L^\varepsilon \varphi^\varepsilon \Rightarrow L^0 \varphi$ at the class $C_0^\infty(\mathbf{R}^n \times \Theta)$ when $\varepsilon \rightarrow 0$. To prove the weak convergence we should show relative compactness of the family $(\tilde{\xi}_t^\varepsilon, \theta_t^\varepsilon)$ in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$. To do this, we use Lemma 2.

Let's study the operator (4) at the test-function $\varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varphi(x_1, \dots, x_n) + \varepsilon \varphi_1(x_1, \dots, x_n; \theta)$, where $\varphi_1(x_1, \dots, x_n; \theta) = -R_0 c(\theta) S(\theta) \varphi(x_1, \dots, x_n)$.

We have:

$$L^\varepsilon \varphi^\varepsilon(x_1, \dots, x_n; \theta) = \varepsilon^{-2} Q \varphi(x_1, \dots, x_n) + \varepsilon^{-1} [Q \varphi_1 + c(\theta) S(\theta) \varphi(x_1, \dots, x_n)] + \\ [c(\theta) S(\theta) \varphi_1(x_1, \dots, x_n; \theta) + c_1(\theta) S(\theta) \varphi(x_1, \dots, x_n; \theta)] + \varepsilon c_1(\theta) S(\theta) \varphi_1(x_1, \dots, x_n; \theta).$$

It follows from (17) that the first two terms equals to 0. Let's estimate the third term:

$$c(\theta) S(\theta) \varphi_1(x_1, \dots, x_n; \theta) + c_1(\theta) S(\theta) \varphi(x_1, \dots, x_n; \theta) = \\ c(\theta) S(\theta) R_0 c(\theta) S(\theta) \varphi(x_1, \dots, x_n) + c_1(\theta) S(\theta) \varphi(x_1, \dots, x_n; \theta) =$$

$$\begin{aligned}
& c^2(\theta)S^2(\theta)\varphi(x_1, \dots, x_n) + c_1(\theta)S(\theta)\varphi(x_1, \dots, x_n; \theta) = \\
& c^2(\theta) \left[\cos^2 \theta_1 \frac{\partial^2}{\partial x_1^2} + \sin^2 \theta_1 \cos^2 \theta_2 \frac{\partial^2}{\partial x_2^2} + \dots + \right. \\
& \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \cos^2 \theta_{n-1} \frac{\partial^2}{\partial x_{n-1}^2} + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin^2 \theta_{n-1} \frac{\partial^2}{\partial x_n^2} + \\
& \left. \left\{ \sin \theta_1 \cos \theta_1 \cos \theta_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \dots + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} \sin \theta_{n-1} \times \right. \right. \\
& \left. \left. \cos \theta_{n-1} \frac{\partial^2}{\partial x_{n-1} \partial x_n} \right\} \right] \varphi(x_1, \dots, x_n) + c_1(\theta) \left[\cos \theta_1 \frac{\partial}{\partial x_1} + \sin \theta_1 \cos \theta_2 \frac{\partial}{\partial x_2} + \dots + \right. \\
& \left. \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1} \frac{\partial}{\partial x_{n-1}} + \sin \theta_1 \dots \sin \theta_{n-2} \sin \theta_{n-1} \frac{\partial}{\partial x_n} \right] \varphi(x_1, \dots, x_n) \leq C_{1,\varphi},
\end{aligned}$$

as soon as all the constants, functions and their derivatives are bounded.

The last term may be estimated analogically.

From (17) we also have:

$$L^\varepsilon \varphi^\varepsilon = L^\varepsilon \varphi + \varepsilon L^\varepsilon \varphi_1 = L^\varepsilon \varphi + \varepsilon L^\varepsilon R_0(\theta)S(\theta)\varphi.$$

Thus,

$$L^\varepsilon \varphi = L^\varepsilon \varphi^\varepsilon - \varepsilon c L^\varepsilon R_0 S(\theta) \varphi \leq C_{1,\varphi} - \varepsilon C_{2,\varphi} < C_\varphi$$

for ε that are small enough.

To prove the second condition of Lemma 2, we use the following properties of the function $\varphi_0(u) = \sqrt{1+u^2}$:

$$|\varphi'_0(u)| \leq 1 \leq \varphi_0(u), |\varphi''_0(u)| \leq \varphi_0(u).$$

So, the proof of the second condition is similar to the previous reasoning.

Thus, the family of the processes $(\tilde{\xi}_t^\varepsilon, \theta_t^\varepsilon)$ is relatively compact in $\mathbf{D}_{\mathbf{R}^n \times \Theta}[0, \infty)$.

Using Theorem 2 we confirm the weak convergence in $\mathbf{D}_{\mathbf{R}^n}[0, \infty)$:

$$\tilde{\xi}_t^\varepsilon \Rightarrow \tilde{\xi}_t^0.$$

Really, all the conditions are satisfied. Namely, the family of processes is relatively compact, the generators at the test-functions from the class

$C_0^\infty(\mathbf{R}^n \times \Theta)$ converge, initial conditions for the limit and pre-limit processes are equal.

Theorem is proved.

Example 3: We study one more variant of evolution in \mathbf{R}^2 .

Let

$$c(\theta) = \begin{cases} 1, \theta = 0, \\ 1, \theta = \pi, \end{cases}$$

and

$$c_1(\theta) = 1, \theta = \frac{\pi}{2}.$$

In other cases both functions equal to 0.

The balance condition (12) is true for $c(\theta)$, on the contrary, condition (13) is true for $c_1(\theta)$.

The limit generator (14) has the view

$$L^0\varphi(x_1, x_2) = \frac{1}{2\pi} \frac{\partial}{\partial x_2} \varphi(x_1, x_2) + \frac{1}{\pi} \frac{\partial^2}{\partial x_1^2} \varphi(x_1, x_2).$$

Thus, the limit process has two parts - the drift with velocity $\frac{1}{2\pi}$ in direction of x_2 coordinate and diffusion part similar to the limit process in M.Kac model [8] in one-dimensional subspace, corresponding to x_1 coordinate.

References

1. Anisimov V. V. Switching processes in queueing models. — Applied Stochastic Methods Series, ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2008 — 345 p.
2. Anisimov V.V. Convergence of accumulation processes with switchings (Ukrainian) // Teor. Prob. Mat. Stat. No. 63 (2000), 3–12; translation in Theory Probab. Math. Statist. No. 63 (2001), 1–11 (2002).
3. Borovkov A.A., Borovkov K. A. Asymptotic analysis of random walks. Heavy-tailed distributions. Encyclopedia of Mathematics and its Applications, 118. — Cambridge University Press, Cambridge, 2008 — 625 p.

4. Borovkov K. On random walks with jumps scaled by cumulative sums of random variables // Statist. Probab. Lett. — 1997. — 35, no. 4. — p. 409–416.
5. Ethier S.N., Kurtz T.G. Markov Processes: Characterization and convergence. — New York: J. Wiley & Sons, 1986 — 534 p.
6. Foss S., Konstantopoulos T., Zachary S. Discrete and continuous time modulated random walks with heavy-tailed increments // J. Theoret. Probab. — 2007. — 20, no. 3. — p. 581–612.
7. Zakhari S., Foss S. G. On the exact asymptotics of the maximum of a random walk with increments in a class of light-tailed distributions // Siberian Math. J. — 2006. — 47, no. 6. — p. 1034–1041.
8. Kac M. A stochastic model related to the telegrapher's equation // Rocky Mountain Math. Journal 4, 497-510 (1974).
9. Kolesnik A.D., Turbin A.F. Infinitesimal hyperbolic operator of Markovian random evolutions // Dokl Acad. Nauk Ukraine, 1, 11-14 (1991). (in Russian).
10. Kolesnik A.D., Turbin A.F. Symmetric random evolutions in \mathbf{R}^2 // Dokl Acad. Nauk Ukraine, 2, 10-11 (1990). (in Russian)
11. Kolesnik A.D. Weak convergence of a planar random evolution to the Wiener process // J. Theoret. Probab. — 14 (2001). — no. 2. — p. 485–494.
12. Kolesnik A.D. Weak convergence of the distributions of Markovian random evolutions in two and three dimensions // Bul. Acad. Stiinte Repub. Mold. Mat. — 2003. — no. 3. — p. 41–52.
13. Koroliuk V.S., Limnios N. Stochastic Systems in Merging Phase Space. — Singapore: World Scientific Publishers, 2005. — 332 p.
14. Orsingher E. Bessel functions of third order and the distribution of cyclic planar motions with three directions // Stoch. Stoch. Rep. 74 (2002), no. 3-4, 617–631.

15. Orsingher, E.; Sommella, A. M. A cyclic random motion in \mathbf{R}^3 with four directions and finite velocity // Stoch. Stoch. Rep. — 76 (2004). — no. 2. — p. 113–133.
16. Pinsky M. Lectures on random evolutions. — Singapore: World Scientific, 1991. — 136 p.
17. Samoilenko I.V. Asymptotic expansion for the functional of Markovian evolution in \mathbf{R}^d in the circuit of diffusion approximation // J. Appl. Math. Stoch. Anal. — 2005. — no. 3. — p. 247–257.
18. Samoilenko I.V. Markovian random evolution in \mathbf{R}^n // Rand. Operat. and Stoc. Equat. — 2001. — №2. — p. 139-160.
19. Samoilenko I.V. Convergence of an impulsive storage process with jump switchings (in Ukrainian) // Ukrain. Mat. Zh. 60 (2008), no. 9, 1282–1286; translation in Ukrainian Math. J. 60 (2008), no. 9, 1492-1497.
20. Stroock D.W., Varadhan S.R.S. Multidimensional Diffusion Processes. — Berlin: Springer-Verlag, 1979. — 338 p.